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Economic order quantity models for ameliorating/deteriorating items under inflation and time discounting

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Abstract

The items that incur a gradual loss in quality or quantity over time while in inventory are usually called deteriorating items. In reality, there are some items whose value or utility or quantity increase with time and those items can be termed as ameliorating items. In this paper, an effort has been made to incorporate these two opposite physical characteristics of stored items into inventory model. We develop models for ameliorating/deteriorating items with time-varying demand pattern over a finite planning horizon, taking into account the effects of inflation and time value of money. Optimal solutions of the proposed models are derived and the effects of amelioration/deterioration on the inventory replenishment policies are studied with the help of numerical examples.

Keywords: Inventory; Time-varying demand; Deterioration; Amelioration; Inflation; Time discounting

1. Introduction

In the past decades, the replenishment scheduling problems were typically attacked by developing proper mathematical models that consider practical factors in real world situations, such as non-stationary demand, physical characteristics of inventoried goods, effects of inflation and time value of money, partial backlogging of unsatisfied demand, etc. It is usually observed in the marketplace that the demand for inventory items increases with time in the *growth phase*, and decreases in the *decline phase*. So researchers commonly use a time-varying demand pattern to reflect sales in different phases of product life cycle. In the early 1970s, Silver and Meal [27] derived an approximate solution procedure for the general case of a deterministic, time-varying demand pattern. Donaldson [12] then considered an inventory model with a

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linear trend in demand. After Donaldson [12], numerous research works have been carried out incorporating time-varying demand into inventory models under a variety of circumstances.

The assumption that the goods in inventory always preserve their physical characteristics is not true in general because there are some items which are subject to risks of breakage, evaporation, obsolescence etc. Decay, change or spoilage that prevent the items from being used for its original purpose are usually termed as deterioration. Food items, pharmaceuticals, photographic film, chemicals and radioactive substances, to name only a few items in which appreciable deterioration can take place during the normal storage of the units. The first attempt to obtain optimal replenishment policies for deteriorating items was made by Ghare and Schrader [13], who derived a revised form of the economic order quantity (EOQ) model assuming exponential decay. Thereafter, a great deal of research efforts have been devoted to inventory models of deteriorating items, the details can be found in the review articles by Raafat [25] and Goyal and Giri [14].

One of the assumptions in most derivations of the inventory model has been a negligible level of inflation. But in recent times many countries have been confronted with fluctuating inflation rates that often have been far from negligible [22]. Silver et al. [28] investigated the impact of inflation on the choice of replenishment quantities in the basic EOQ model. The pioneer in this field was Buzacott [5], who developed the first EOQ model taking inflation into account. Several researchers have extended their approach to various interesting situations by considering the time value of money, different inflation rates for the internal and external costs, finite replenishment rate, shortages, etc. The models of Mishra [21], Bierman and Thomas [2], Aggarwal [1], Chandra and Bahner [6], Hariga and Ben-Daya [17], Ray and Chaudhuri [26], Mangiameli et al. [20], Brahmbhatt [4], Dohi et al. [11], Moon and Yun [24], and Moon and Lee [23] are worth mentioning in this direction.

Bose et al. [3] first explored a deteriorating inventory model under inflation and time value of money. Unfortunately, their model contains some mathematical errors in the formulation of the holding cost and the purchase cost which lead to incorrect total cost function [22]. Chen [8] proposed a generalized dynamic programming model over a finite planning horizon for items with Weibull distribution deterioration where the demand rate is assumed to be time-proportional, shortages are allowed and are completely backordered and the effects of inflation and time value of money are taken into consideration. This model permits variation in both the replenishment intervals and the service levels between the order cycles. Chung et al. [10] discussed the inventory replenishment policy over a finite planning horizon for a deteriorating item taking account of time value and presented a line search technique to decide the optimal interval which has positive inventories. Recently, Chung and Lin [9] extended the inventory replenishment model of Chung et al. [10] to the situation where shortages are allowed in each replenishment cycle (the model starts with inventory, and ends with shortages). They considered the demand rate to be known and constant and applied the DCF approach to determine the optimal number of replenishments and the corresponding cycle length, consisting of positive and negative inventories.

Although degradation (or loss) of value or utility or quantity of some physical goods is a common experience in reality, there are some items whose value or utility increase over time by ameliorating activation, e.g. wine. It is a practical experience in wine manufacturing industry that utility or value of some kind of wine increases by age. Other examples can be high breed fishes in breeding yard (fish culture facility) or fast growing animals like broiler, pig, etc. in farming yard. In this paper, the term "amelioration" means to make better or increase goods in quantity or amount in an inventory. Hwang [18,19] for the first time developed EOQ models for items which are ameliorating in nature. In the present article, our attempt is to incorporate both the opposite physical characteristics viz. amelioration and deterioration of stored items into inventory model. We develop models with zero-ending inventory for fixed order intervals over a finite planning horizon allowing (A) shortages in all but not in the last cycle and (B) shortages in all cycles [15], taking into account the effects of inflation and time value of money. Optimal solutions of the proposed models are derived and the effects of amelioration/deterioration on the inventory replenishment policies are studied with the help of numerical examples.

2. Assumptions and notation

The following assumptions and notation are used in developing the models:

(a) H is taken to be the fixed time horizon.
(b) n orders are placed during the time horizon H and the replenishment rate is infinite i.e. replenishment is instantaneous.
(c) Shortages are allowed to occur in inventory and lead time is zero.
(d) The demand $f(t)$ at time t is a continuous function of time.
(e) A constant fraction $\alpha(0 \leqslant \alpha < 1)$ of on-hand inventory ameliorates or deteriorates per unit of time.
(f) r is the discount rate representing the time value of money.
(g) At time $t = 0$, c_{11} and c_{12} are internal and external holding costs per unit item per unit time; c_{21} and c_{22} are internal and external shortage costs per unit item per unit time.
(h) At time $t = 0$, A is the fixed internal ordering cost per order and p is the external purchase cost.
(i) Internal and external inflation rates are denoted by i_1 and i_2, respectively.
(j) C_o, C_h, C_s and C_p are, respectively, the present worth of total replenishment cost, total inventory holding cost, total shortage cost and total purchase cost during the fixed time horizon H.

We also assume that $r > i_1, i_2$, i.e. the interest rate is larger than the internal and external inflation rates, which is a practical assumption.

3. Model development

3.1. Model A with shortages in all but not in the last cycle

We assume that the replenishment cycles are of equal length and the no-shortage period in each cycle is a constant fraction $K(0 < K < 1)$ of each replenishment interval. Let $t_j = (j-1)H/n$ be the time of the jth replenishment, $j = 1, 2, \dots, n$ and $s_j = t_j + KH/n = (K + j - 1)H/n$ be the time at which the inventory level in the jth replenishment cycle drops to zero, $j = 1, 2, \dots, n-1$ and $s_n = H$.

3.1.1. Formulation of the basic model

The amount of ameliorated or deteriorated units during a given time interval depends on the on-hand inventory and the elapsed time in the system. Therefore, if $I_1(t)$ denotes the inventory level at any time t in the jth replenishment cycle in $[t_j, s_j]$, $j = 1, 2, \dots, n$ during the period of positive inventory, then the instantaneous state of $I_1(t)$ can be described by the following differential equation:

$$\frac{\mathrm{d}I_1(t)}{\mathrm{d}t} = -f(t) - \alpha_\delta I_1(t), \quad t_j \leqslant t \leqslant s_j, \quad j = 1, 2, \dots, n \tag{1}$$

with the boundary condition $I_1(s_j) = 0$, where

$$\alpha_\delta = \begin{cases} \alpha & \text{in case of deteriorating items,} \\ -\alpha & \text{in case of ameliorating items.} \end{cases}$$

The differential equation governing the system during the period of shortage is given by

$$\frac{\mathrm{d}I_2(t)}{\mathrm{d}t} = f(t), \quad s_j \leqslant t \leqslant t_{j+1}, \quad j = 1, 2, \dots, n-1 \tag{2}$$

with the initial condition $I_2(s_j) = 0$, where $I_2(t)$ is the shortage level at time t in the jth replenishment cycle, $j = 1, 2, \dots, n-1$.

The solutions of the differential equations (1) and (2) are given by

$$I_1(t) = \int_t^{s_j} e^{\alpha_\delta(u-t)} f(u)\,du, \quad t_j \leqslant t \leqslant s_j, \quad j = 1,2,\ldots,n, \tag{3}$$

$$I_2(t) = \int_{s_j}^{t} f(u)\,du, \quad s_j \leqslant t \leqslant t_{j+1}, \quad j = 1,2,\ldots,n-1. \tag{4}$$

The total inventory cost, during the fixed time horizon H, consists of replenishment or ordering cost, holding cost, shortage cost, and purchase cost.

The present worth of the total replenishment cost during H is given by

$$C_o = A\sum_{j=1}^{n} e^{-R_1 t_j} = A\sum_{j=1}^{n} e^{-R_1(j-1)H/n} = \frac{A(1-e^{-R_1 H})}{1-e^{-R_1 H/n}}, \quad \text{where } R_1 = r - i_1.$$

The present worth of the total inventory holding cost during H is

$$C_h = \sum_{j=1}^{n}\sum_{m=1}^{2} c_{1m}\int_{t_j}^{s_j} I_1(t)e^{-R_m t}\,dt = \sum_{j=1}^{n}\sum_{m=1}^{2} c_{1m}\int_{t_j}^{s_j}\left[\int_t^{s_j} e^{\alpha_\delta(u-t)} f(u)\,du\right]e^{-R_m t}\,dt$$
$$= \sum_{j=1}^{n}\sum_{m=1}^{2} \frac{c_{1m}}{R_m+\alpha_\delta}\int_{t_j}^{s_j}\left\{e^{\alpha_\delta(t-t_j)-R_m t_j} - e^{-R_m t}\right\}f(t)\,dt, \quad \text{where } R_m = r - i_m, m = 1,2.$$

Similarly, the present worth of the total shortage cost during H is

$$C_s = \sum_{j=1}^{n-1}\sum_{m=1}^{2} c_{2m}\int_{s_j}^{t_{j+1}} I_2(t)e^{-R_m t}\,dt = \sum_{j=1}^{n-1}\sum_{m=1}^{2} c_{2m}\int_{s_j}^{t_{j+1}}\left[\int_{s_j}^{t} f(u)\,du\right]e^{-R_m t}\,dt$$
$$= \sum_{j=1}^{n-1}\sum_{m=1}^{2} \frac{c_{2m}}{R_m}\int_{s_j}^{t_{j+1}}\left(e^{-R_m t} - e^{-R_m t_{j+1}}\right)f(t)\,dt.$$

And the present worth of the total purchase cost during H is

$$C_p = p\left[\sum_{j=1}^{n} I_1(t_j)e^{-R_2 t_j} + \sum_{j=1}^{n-1} I_2(t_{j+1})e^{-R_2 t_{j+1}}\right]$$
$$= p\left[\sum_{j=1}^{n} e^{-R_2 t_j}\int_{t_j}^{s_j} e^{\alpha_\delta(t-t_j)} f(t)\,dt + \sum_{j=1}^{n-1} e^{-R_2 t_{j+1}}\int_{s_j}^{t_{j+1}} f(t)\,dt\right].$$

Hence, the present worth of the total variable cost of the inventory system during the entire time period H is given by

$$\begin{aligned} \mathrm{TC}(n,K) = C_o + C_h + C_s + C_p = \frac{A(1-e^{-R_1 H})}{1-e^{-R_1 H/n}} + \sum_{j=1}^{n}\sum_{m=1}^{2}\frac{c_{1m}}{R_m+\alpha_\delta}\int_{t_j}^{s_j}\left\{e^{\alpha_\delta(t-t_j)-R_m t_j} - e^{-R_m t}\right\}f(t)\,dt \\ + \sum_{j=1}^{n-1}\sum_{m=1}^{2}\frac{c_{2m}}{R_m}\int_{s_j}^{t_{j+1}}\left(e^{-R_m t} - e^{-R_m t_{j+1}}\right)f(t)\,dt + p\left[\sum_{j=1}^{n} e^{-R_2 t_j}\int_{t_j}^{s_j} e^{\alpha_\delta(t-t_j)} f(t)\,dt\right. \\ \left. + \sum_{j=1}^{n-1} e^{-R_2 t_{j+1}}\int_{s_j}^{t_{j+1}} f(t)\,dt\right], \quad \text{where } t_j = (j-1)H/n,\ s_j = (K+j-1)H/n, \\ R_m = r - i_m,\ m = 1,2. \end{aligned} \tag{5}$$

Our objective is to determine the optimal values of n and K that minimize $\mathrm{TC}(n,K)$.

3.1.2. Solution procedure

Since $\mathrm{TC}(n,K)$ is a function of a discrete variable n and a continuous variable K $(0<K<1)$, therefore, for any given n, the necessary condition for the minimum of $\mathrm{TC}(n,K)$ is

$$\frac{\mathrm{dTC}(n,K)}{\mathrm{d}K}=0,$$

which gives

$$G(n,K)\equiv\sum_{j=1}^{n-1}\sum_{m=1}^{2}\frac{c_{1m}}{R_m+\alpha_\delta}\left[\mathrm{e}^{\alpha_\delta\frac{H}{n}K-R_m(j-1)\frac{H}{n}}-\mathrm{e}^{-R_m\frac{H}{n}(K+j-1)}\right]$$
$$-\sum_{j=1}^{n-1}\sum_{m=1}^{2}\frac{c_{2m}}{R_m}\left[\mathrm{e}^{-R_m\frac{H}{n}(K+j-1)}-\mathrm{e}^{-R_m\frac{jH}{n}}\right]+p\sum_{j=1}^{n-1}\left[\mathrm{e}^{\alpha_\delta\frac{H}{n}K-R_2(j-1)\frac{H}{n}}-\mathrm{e}^{-R_2\frac{jH}{n}}\right]=0. \tag{6}$$

Case (i) $\alpha_\delta=\alpha$

Proposition 1. *For any given $n\geqslant 2$, there exists a solution of Eq.* (6) *provided the condition* (7) *given below is satisfied.*

Proof. Clearly $G(n,1)$ is strictly positive for a given value of n. In order to guarantee the existence of a solution $K^*(0<K^*<1)$ of Eq. (6), $G(n,0)$ should be strictly negative and that the condition can be simplified to the following:

$$\left(p-\frac{c_{22}}{R_2}\right)\left(1-\mathrm{e}^{-R_2\frac{n-1}{n}H}\right)<\frac{c_{21}}{R_1}\left(1-\mathrm{e}^{-R_1\frac{n-1}{n}H}\right). \tag{7}$$

Especially, when $p<\frac{c_{22}}{R_2}$, the above condition is always satisfied. The possibility of satisfying this condition increases when $r\to i_2$. □

Proposition 2. *If there exists a solution of Eq.* (6) *then it is the unique global optimal solution.*

Proof. Differentiating $G(n,K)$ with respect to K we get

$$\frac{\mathrm{d}G(n,K)}{\mathrm{d}K}=\frac{H}{n}\left[\sum_{j=1}^{n-1}\sum_{m=1}^{2}\frac{c_{1m}}{R_m+\alpha}\left\{\alpha\,\mathrm{e}^{\alpha\frac{H}{n}K-R_m(j-1)\frac{H}{n}}+R_m\mathrm{e}^{-R_m\frac{H}{n}(K+j-1)}\right\}\right.$$
$$\left.+\sum_{j=1}^{n-1}\sum_{m=1}^{2}c_{2m}\,\mathrm{e}^{-R_m\frac{H}{n}(K+j-1)}+p\alpha\sum_{j=1}^{n-1}\mathrm{e}^{-R_2(j-1)\frac{H}{n}+\alpha\frac{H}{n}K}\right]>0,$$

since $R_m+\alpha\geqslant 0,\ m=1,2$ and $K\in(0,1]$.

Therefore, for any given $n\geqslant 2$, $G(n,K)$ is a strictly increasing function of K and consequently the solution K^* of Eq. (6) is unique. Moreover, it can be easily verified that the second-order derivative $\frac{\mathrm{d}^2\mathrm{TC}(n,K)}{\mathrm{d}K^2}\,|_{K=K^*}$ is strictly positive. Hence the proof is complete. □

Remarks. (i) When $\alpha_\delta=\alpha$ the proposed model is identical to Bose et al. [3] and moreover, if there is no inflation and time discounting, the model reduces to that of Hariga [16]. Unfortunately, Bose et al.'s [3] model contains mathematical errors in the formulation of the holding cost and purchase cost, as pointed out by Moon and Giri [22]. (ii) It is clear from Eq. (6) that the optimal value of K does not depend on the demand function parameters which is in contrast to the value obtained by Bose et al. [3].

Case (*ii*) $\alpha_\delta = -\alpha$

Proposition 3. *If* $p < \frac{c_{22}}{R_2}$ *and* $R_m - \alpha \geqslant 0$ *for* $m = 1, 2$. *then there exists at least a local optimal solution of Eq.* (6).

Proof. If $R_m - \alpha \geqslant 0$ for $m = 1, 2$ then by (7) clearly the condition $p < \frac{c_{22}}{R_2}$ guarantees the existence of a solution K^* in Eq. (6). It is not straightforward to show that for any given n, $G(n, K)$ is an increasing function of K under the conditions stated above. However, after a little simplification, the second-order derivative of $\mathrm{TC}(n, K)$ at $K = K^*$ can be obtained as

$$\frac{\mathrm{d}^2\mathrm{TC}(n,K)}{\mathrm{d}K^2}\Big|_{K=K^*} = \sum_{j=1}^{n-1}\sum_{m=1}^{2}(c_{1m}+c_{2m})\mathrm{e}^{-R_m\frac{H}{n}(K^*+j-1)} - \alpha\left[\sum_{j=1}^{n-1}\sum_{m=1}^{2}\frac{c_{2m}}{R_m}\left\{\mathrm{e}^{-R_m\frac{H}{n}(K^*+j-1)} - \mathrm{e}^{-R_m\frac{jH}{n}}\right\} + p\sum_{j=1}^{n-1}\mathrm{e}^{-R_2\frac{jH}{n}}\right] > 0, \tag{8}$$

when $p < \frac{c_{22}}{R_2}$ and $R_m - \alpha \geqslant 0$, $m = 1, 2$. This proves that K^* is a local minimizer of $\mathrm{TC}(n, K)$.

When $R_m - \alpha < 0$ for $m = 1, 2$ or $R_1 - \alpha > 0$, $R_2 - \alpha < 0$ (the situation $R_1 - \alpha < 0$, $R_2 - \alpha > 0$ is not possible since $i_2 > i_1$ in general) the condition $G(n, 0)G(n, 1) < 0$ must hold for the existence of a solution K^* in (6) and condition (8) for verifying it as a minimizing solution. In any case, if a solution of Eq. (6) exists, it can be obtained numerically by any one-dimensional search technique. We outline below a simple algorithm based on line search technique. □

Algorithm

Step 1. Set $n = 1$, $\mathrm{TC}^0 =$ a very large positive value.
Step 2. Set $n = n + 1$, $K_{\mathrm{low}} = 0$ and $K_{\mathrm{high}} = 1$.
Step 3. Set $K^* = (K_{\mathrm{low}} + K_{\mathrm{high}})/2$.
Step 4. If $G(n, K^*) = 0$, go to Step 6. Otherwise, go to Step 5.
Step 5. If $G(n, K^*) > 0$, set $K_{\mathrm{high}} = K^*$ and go to Step 3.
If $G(n, K^*) < 0$, set $K_{\mathrm{low}} = K^*$ and go to Step 3.
Step 6. Compute $t_j = (j-1)H/n$, $j = 1, 2, \ldots, n$; $s_j = (K^* + j - 1)H/n$, $j = 1, 2, \ldots, n-1$ and then $\mathrm{TC}(n, K^*)$ from (5).
Step 7. If the condition (8) is satisfied, then go to Step 8. Otherwise, K^* is not a minimizing solution. Stop.
Step 8. If $\mathrm{TC}^0 \geqslant \mathrm{TC}(n, K^*)$, assign $\mathrm{TC}^0 = \mathrm{TC}(n, K^*)$ and go to Step 2. Otherwise, the minimum total cost is TC^0, $n^* = n - 1$ and the corresponding value of K^* is the required solution.

3.2. Extended cases

3.2.1. Model A1 with unequal periods of positive inventory

We now extend the basic Model A by relaxing the assumption of equal no-shortage interval in each cycle. Let K_j be the fraction of the jth replenishment interval ($j = 1, 2, \ldots, n-1$) for which there is no-shortage in inventory. Then the shortage points s_j's would be $s_j = t_j + K_jH/n = (K_j + j - 1)H/n$, $0 < K_j < 1$, $j = 1, 2, \ldots, n-1$.

As a result, the necessary conditions for the minimum of the total variable cost $\mathrm{TC}(n; K_1, K_2, \ldots, K_{n-1})$ give

$$G_{1j}(n,K_j) \equiv \sum_{m=1}^{2} \frac{c_{1m}}{R_m+\alpha_\delta}\left[\mathrm{e}^{\alpha_\delta \frac{H}{n}K_j - R_m(j-1)\frac{H}{n}} - \mathrm{e}^{-R_m\frac{H}{n}(K_j+j-1)}\right] - \sum_{m=1}^{2}\frac{c_{2m}}{R_m}\left[\mathrm{e}^{-R_m\frac{H}{n}(K_j+j-1)} - \mathrm{e}^{-R_m\frac{jH}{n}}\right]$$
$$+\,p\left[\mathrm{e}^{\alpha_\delta \frac{H}{n}K_j - R_2(j-1)\frac{H}{n}} - \mathrm{e}^{-R_2\frac{jH}{n}}\right] = 0, \quad j=1,2,\ldots,n-1. \tag{9}$$

The above equations show that the value of j influences the value of K in each replenishment cycle. Similar to the previous section, the existence and uniqueness of K_j's, $j=1,2,\ldots,n-1$ can be examined when $\alpha_\delta=\alpha$ or $-\alpha$. The condition (7), in this case, would take the form

$$\left(p-\frac{c_{22}}{R_2}\right)(1-\mathrm{e}^{-R_2H/n})\mathrm{e}^{-R_2(j-1)H/n} < \frac{c_{21}}{R_1}(1-\mathrm{e}^{-R_1H/n})\mathrm{e}^{-R_1(j-1)H/n} \quad j=1,2,\ldots,n-1. \tag{10}$$

3.2.1.1. Model A1 with partial backlogging. The backlogging rate in many inventory systems depends on the length of the waiting time for the next replenishment. The longer the waiting time is, the smaller the backlogging rate would be. The proportion of customers who would like to accept backorder at time t decreases with the waiting time $(t_{j+1}-t)$ before the next replenishment. Considering this situation, the backlogging rate can be defined as

$$\text{backlogging rate} = \frac{1}{1+\beta(t_{j+1}-t)}, \quad s_j \leqslant t \leqslant t_{j+1},$$

where $\beta(\geqslant 0)$ is called the backlogging parameter.

There are relevant particulars related to the above definition in Chang and Dye [7].

In this case, the differential equation governing the system during the period of shortages is given by

$$\frac{\mathrm{d}I_2(t)}{\mathrm{d}t} = \frac{f(t)}{1+\beta(t_{j+1}-t)}, \quad s_j \leqslant t \leqslant t_{j+1} \tag{11}$$

with the initial condition $I_2(s_j)=0$, where $I_2(t)$ is the shortage level at time t.

Solving (11) we get

$$I_2(t) = \int_{s_j}^{t} \frac{f(u)}{1+\beta(t_{j+1}-u)}\,\mathrm{d}u, \quad s_j \leqslant t \leqslant t_{j+1}. \tag{12}$$

Then the present worth of the total shortage cost during H is

$$C_s = \sum_{j=1}^{n-1}\sum_{m=1}^{2} c_{2m}\int_{s_j}^{t_{j+1}} I_2(t)\mathrm{e}^{-R_m t}\,\mathrm{d}t = \sum_{j=1}^{n-1}\sum_{m=1}^{2} c_{2m}\int_{s_j}^{t_{j+1}}\left[\int_{s_j}^{t}\frac{f(u)}{1+\beta(t_{j+1}-u)}\,\mathrm{d}u\right]\mathrm{e}^{-R_m t}\,\mathrm{d}t$$
$$= \sum_{j=1}^{n-1}\sum_{m=1}^{2}\frac{c_{2m}}{R_m}\int_{s_j}^{t_{j+1}}\frac{\mathrm{e}^{-R_m t}-\mathrm{e}^{-R_m t_{j+1}}}{1+\beta(t_{j+1}-t)}f(t)\,\mathrm{d}t.$$

If c_{31} and c_{32} are internal and external opportunity costs due to lost sales per unit item per unit time, the present worth of the total lost sales during H is given by

$$C_l = \sum_{j=1}^{n-1}\sum_{m=1}^{2} c_{3m}\int_{s_j}^{t_{j+1}}\left[1-\frac{1}{1+\beta(t_{j+1}-t)}\right]f(t)\mathrm{e}^{-R_m t}\,\mathrm{d}t = \beta\sum_{j=1}^{n-1}\sum_{m=1}^{2} c_{3m}\int_{s_j}^{t_{j+1}}\frac{(t_{j+1}-t)\mathrm{e}^{-R_m t}}{1+\beta(t_{j+1}-t)}f(t)\,\mathrm{d}t.$$

Also the present worth of the total purchase cost during H becomes

$$\begin{aligned} C_{\mathrm{p}} &= p\left[\sum_{j=1}^{n} I_1(t_j)\mathrm{e}^{-R_2 t_j} + \sum_{j=1}^{n-1} I_2(t_{j+1})\mathrm{e}^{-R_2 t_{j+1}}\right] \\ &= p\left[\sum_{j=1}^{n} \mathrm{e}^{-R_2 t_j}\int_{t_j}^{s_j} \mathrm{e}^{\alpha_\delta(t-t_j)} f(t)\,\mathrm{d}t + \sum_{j=1}^{n-1} \mathrm{e}^{-R_2 t_{j+1}}\int_{s_j}^{t_{j+1}} \frac{f(t)}{1+\beta(t_{j+1}-t)}\,\mathrm{d}t\right]. \end{aligned}$$

We can formulate the present worth of the total inventory cost as the sum of the replenishment cost, holding cost, shortage cost, opportunity cost due to lost sales, and purchase cost. Therefore,

$$\begin{aligned} \mathrm{TC}(n,\{K_j\}) &= C_{\mathrm{o}} + C_{\mathrm{h}} + C_{\mathrm{s}} + C_{\mathrm{l}} + C_{\mathrm{p}} \\ &= \frac{A(1-\mathrm{e}^{-R_1 H})}{1-\mathrm{e}^{-R_1 H/n}} + \sum_{j=1}^{n}\sum_{m=1}^{2} \frac{c_{1m}}{R_m+\alpha_\delta}\int_{t_j}^{s_j} \left\{\mathrm{e}^{\alpha_\delta(t-t_j)-R_m t_j} - \mathrm{e}^{-R_m t}\right\} f(t)\,\mathrm{d}t \\ &\quad + \sum_{j=1}^{n-1}\sum_{m=1}^{2} \frac{c_{2m}}{R_m}\int_{s_j}^{t_{j+1}} \frac{\mathrm{e}^{-R_m t} - \mathrm{e}^{-R_m t_{j+1}}}{1+\beta(t_{j+1}-t)} f(t)\,\mathrm{d}t + \beta\sum_{j=1}^{n-1}\sum_{m=1}^{2} c_{3m}\int_{s_j}^{t_{j+1}} \frac{(t_{j+1}-t)\mathrm{e}^{-R_m t}}{1+\beta(t_{j+1}-t)} f(t)\,\mathrm{d}t \\ &\quad + p\left[\sum_{j=1}^{n} \mathrm{e}^{-R_2 t_j}\int_{t_j}^{s_j} \mathrm{e}^{\alpha_\delta(t-t_j)} f(t)\,\mathrm{d}t + \sum_{j=1}^{n-1} \mathrm{e}^{-R_2 t_{j+1}}\int_{s_j}^{t_{j+1}} \frac{f(t)}{1+\beta(t_{j+1}-t)}\,\mathrm{d}t\right], \end{aligned} \tag{13}$$

where $t_j = (j-1)H/n,\ j = 1,2,\ldots,n;\ s_j = (K_j + j - 1)H/n,\ j = 1,2,\ldots,n-1;\ R_m = r - i_m,\ m = 1,2;$ $\{K_j\} = K_1, K_2, \ldots, K_{n-1}$.

The total variable cost (13) is a function of discrete variable n and continuous variables $K_1, K_2, \ldots, K_{n-1}$. Therefore, for any given n, the values of $K_1, K_2, \ldots, K_{n-1}$ that minimize $\mathrm{TC}(n,\{K_j\})$ can be obtained by solving the following non-linear equations:

$$\begin{aligned} G_{2j}(n,K_j) &\equiv \sum_{m=1}^{2} \frac{c_{1m}}{R_m+\alpha_\delta}\left[\mathrm{e}^{\alpha_\delta \frac{H}{n}K_j - R_m(j-1)\frac{H}{n}} - \mathrm{e}^{-R_m \frac{H}{n}(K_j+j-1)}\right] - n\sum_{m=1}^{2} \frac{c_{2m}}{R_m}\left[\frac{\mathrm{e}^{-R_m \frac{H}{n}(K_j+j-1)} - \mathrm{e}^{-R_m \frac{jH}{n}}}{n+\beta H(1-K_j)}\right] \\ &\quad - \beta\sum_{m=1}^{2} \frac{c_{3m}H(1-K_j)\mathrm{e}^{-R_m \frac{H}{n}(K_j+j-1)}}{n+\beta H(1-K_j)} + p\left[\mathrm{e}^{\alpha_\delta \frac{H}{n}K_j - R_2(j-1)\frac{H}{n}} - \frac{n\mathrm{e}^{-R_2 \frac{jH}{n}}}{n+\beta H(1-K_j)}\right] = 0, \\ & j = 1,2,\ldots,n-1. \end{aligned} \tag{14}$$

Eqs. (14) can be solved for a unique $\{K_j\}$ provided $G_{2j}(n,0)G_{2j}(n,1) < 0$ and $G_{2j}(n,K_j)$ is monotone in $[0,1]$, for each j. When $\alpha_\delta = \alpha$ or $-\alpha$ and $R_m + \alpha_\delta \geqslant 0$ for $m = 1,2$, it is easy to verify that $G_{2j}(n,1)$ is strictly positive and $G_{2j}(n,0)$ would be strictly negative provided n and j satisfy the following relation:

$$\begin{aligned} & n\left(p - \frac{c_{22}}{R_2}\right)\left[\mathrm{e}^{-R_2(j-1)\frac{H}{n}} - \mathrm{e}^{-R_2\frac{jH}{n}}\right] + \beta H(p - c_{32})\mathrm{e}^{-R_2(j-1)\frac{H}{n}} \\ & \quad < \frac{nc_{21}}{R_1}\left[\mathrm{e}^{-R_1(j-1)\frac{H}{n}} - \mathrm{e}^{-R_1\frac{jH}{n}}\right] + \beta H c_{31}\mathrm{e}^{-R_1(j-1)\frac{H}{n}}, \quad j = 1,2,\ldots,n-1. \end{aligned} \tag{15}$$

For uniqueness of the solution, the relevant Hessian matrix (which is a diagonal matrix) should be positive definite implying that $\frac{\partial G_{2j}(n,\{K_j\})}{\partial K_j}$ should be strictly positive for all j. When $\beta = 0$ (complete backlogging), the condition (15) reduces to (10).

3.2.2. Model B with shortages in all cycles

In the previous sections, we have developed inventory models allowing shortages to follow inventory in all but not in the last cycle. Here we allow inventory to follow shortages in every cycle. In order to develop the model, we redefine the stock-out points $s_j = (j-1)H/n$, $j = 1, 2, \ldots, n+1$ and the reorder points $t_j = s_j + K_j H/n = (K_j + j - 1)H/n, j = 1, 2, \ldots, n$, where K_j $(0 < K_j < 1)$ is the proportion of time in the jth replenishment interval during which there are shortages in inventory. The differential equations that describe the instantaneous states of positive inventory and negative inventory (i.e. shortages) are given, respectively by

$$\frac{\mathrm{d}I_1(t)}{\mathrm{d}t} = -f(t) - \alpha_\delta I_1(t), \quad t_j \leqslant t \leqslant s_{j+1}, \quad j = 1, 2, \ldots, n \tag{16}$$

with the boundary condition $I_1(s_{j+1}) = 0$, and

$$\frac{\mathrm{d}I_2(t)}{\mathrm{d}t} = f(t), \quad s_j \leqslant t \leqslant t_j, \quad j = 1, 2, \ldots, n \tag{17}$$

with the initial condition $I_2(s_j) = 0$.

The solutions of the differential equations (16) and (17) are

$$I_1(t) = \int_t^{s_{j+1}} \mathrm{e}^{\alpha_\delta (u-t)} f(u)\,\mathrm{d}u, \quad t_j \leqslant t \leqslant s_{j+1}, \quad j = 1, 2, \ldots, n, \tag{18}$$

$$I_2(t) = \int_{s_j}^{t} f(u)\,\mathrm{d}u, \quad s_j \leqslant t \leqslant t_j, \quad j = 1, 2, \ldots, n. \tag{19}$$

Similar to the previous sections, the present worth of the total variable cost of the system during the entire time period H can be derived as

$$\begin{aligned}\mathrm{TC}(n, \{t_j\}) = {} & A \sum_{j=1}^{n} \mathrm{e}^{-R_1 t_j} + \sum_{j=1}^{n} \sum_{m=1}^{2} \frac{c_{1m}}{R_m + \alpha_\delta} \int_{t_j}^{s_{j+1}} \left\{ \mathrm{e}^{\alpha_\delta (t - t_j) - R_m t_j} - \mathrm{e}^{-R_m t} \right\} f(t)\,\mathrm{d}t \\ & + \sum_{j=1}^{n} \sum_{m=1}^{2} \frac{c_{2m}}{R_m} \int_{s_j}^{t_j} \left(\mathrm{e}^{-R_m t} - \mathrm{e}^{-R_m t_j} \right) f(t)\,\mathrm{d}t + p \sum_{j=1}^{n} \mathrm{e}^{-R_2 t_j} \left[\int_{t_j}^{s_{j+1}} \mathrm{e}^{\alpha_\delta (t - t_j)} f(t)\,\mathrm{d}t \right. \\ & \left. + \int_{s_j}^{t_j} f(t)\,\mathrm{d}t \right] \quad \text{where } \{t_j\} = t_1, t_2, \ldots, t_n.\end{aligned} \tag{20}$$

For any given n, the necessary criteria for the minimum of $\mathrm{TC}(n, \{t_j\})$ yield

$$\begin{aligned}F(n, t_j) \equiv {} & -AR_1 \mathrm{e}^{-R_1 t_j} - \sum_{m=1}^{2} c_{1m} \int_{t_j}^{s_{j+1}} \mathrm{e}^{\alpha_\delta (t - t_j) - R_m t_j} f(t)\,\mathrm{d}t \\ & + \sum_{m=1}^{2} c_{2m} \int_{s_j}^{t_j} \mathrm{e}^{-R_m t_j} f(t)\,\mathrm{d}t - p(R_2 + \alpha_\delta) \int_{t_j}^{s_{j+1}} \mathrm{e}^{\alpha_\delta (t - t_j) - R_2 t_j} f(t)\,\mathrm{d}t - pR_2 \int_{s_j}^{t_j} \mathrm{e}^{-R_2 t_j} f(t)\,\mathrm{d}t = 0, \\ & j = 1, 2, \ldots\ldots, n.\end{aligned} \tag{21}$$

We see from above that the optimal values of t_j and hence $K_j (j = 1, 2, \ldots, n)$ are dependent on the demand function parameters. When $\alpha_\delta = \alpha$ or $-\alpha$ and $R_2 + \alpha_\delta \geqslant 0$, $F(n, s_j) < 0$ and therefore, for the existence of t_j's in Eq. (21) the following condition must hold:

$$\begin{aligned}F\left(n, s_j + \frac{H}{n}\right) = {} & -AR_1 \mathrm{e}^{-R_1 \frac{jH}{n}} + \sum_{m=1}^{2} c_{2m} \int_{(j-1)\frac{H}{n}}^{jH/n} \mathrm{e}^{-R_m \frac{jH}{n}} f(t)\,\mathrm{d}t - pR_2 \int_{(j-1)\frac{H}{n}}^{jH/n} \mathrm{e}^{-R_2 \frac{jH}{n}} f(t)\,\mathrm{d}t > 0, \\ & j = 1, 2, \ldots, n.\end{aligned}$$

Simplifying, we get

$$\left(p-\frac{c_{22}}{R_2}\right)\mathrm{e}^{-R_2\frac{jH}{n}}\int_{(j-1)\frac{H}{n}}^{\frac{jH}{n}} f(t)\,\mathrm{d}t<\frac{R_1}{R_2}\mathrm{e}^{-R_1\frac{jH}{n}}\left[\frac{c_{21}}{R_1}\int_{(j-1)\frac{H}{n}}^{jH/n} f(t)\,\mathrm{d}t-A\right]\quad j=1,2,\ldots,n.$$

It is not easy to verify analytically the existence of a solution $\{t_j\}$ in (21) when $\alpha_\delta=-\alpha$, $R_1-\alpha>0$, $R_2-\alpha<0$ or when $R_m-\alpha<0$, $m=1,2$. In such a case, any commercial software package would be useful to obtain the optimal solution numerically.

4. Numerical examples

Example 1. For numerical experiment, we choose the following data taken from Bose et al. [3].

$f(t)=20+50t,\ p=5,\ c_{11}=0.2,\ c_{12}=0.4,\ c_{21}=0.8,\ c_{22}=0.6,\ A=80,\ r=0.2,\ i_1=0.08,\ i_2=0.14,$ $H=10$, $\alpha=0.01$ (in appropriate units).

Using these parameter values, the optimal solution of the basic Model A for deteriorating items ($\alpha_\delta=\alpha$) is obtained as $n^*=13$, $K^*=0.497381$ and $\mathrm{TC}(n^*,K^*)=17219.14$. Bose et al. [3] found $n^*=15$, $K^*=0.288911$ and $\mathrm{TC}(n^*,K^*)=17648.09$. This means that 2.43% overestimation in the total cost occurs in their model due to incorrect formulation of the total cost function. The optimal solution of the basic model for ameliorating items ($\alpha_\delta=-\alpha$) is $n^*=13$, $K^*=0.527385$ and $\mathrm{TC}(n^*,K^*)=17177.92$. Tables 1 and 2 show the values of K_j^*'s that minimize the total cost functions in Model A1 and Model B, respectively.

A comparison of the output of the proposed models is shown in Table 3. Clearly the total cost in Model A1 is less than that of Model A, but the difference is negligible while Model B provides the least cost, as expected.

It is also noted that as the rate of amelioration increases, total costs in Models A, A1 and B decrease monotonically. The effects of amelioration on the total costs can be observed from Table 4. # indicates the result when the optimal values of n and K or K_j's for $\alpha=0$ are substituted in the objective function. To examine the outcome of Model A1 with partial backlogging, we take $c_{31}=1.0$, $c_{32}=0.8$ and $\beta=0.1$ in addition to the other parameter values already mentioned at the beginning of this section. The total costs for deteriorating and ameliorating items reduces to 17082.26 and 17051.52, respectively. This indicates that the partial backlogging option, in the present situation, is highly desirable over complete backlogging from management's point of view.

Table 1
The values of K_j^*'s in Model A1

j	K_j^*		
	$\alpha_\delta=-\alpha$	$\alpha_\delta=0$	$\alpha_\delta=\alpha$
1	0.553284	0.538589	0.524629
2	0.547739	0.532873	0.518769
3	0.542198	0.527170	0.512928
4	0.536667	0.521484	0.507112
5	0.531152	0.515821	0.501326
6	0.525660	0.510188	0.495576
7	0.520195	0.504588	0.489869
8	0.514763	0.499031	0.484209
9	0.509369	0.493519	0.478602
10	0.504019	0.488057	0.473052
11	0.498718	0.482651	0.467566
12	0.493471	0.477307	0.462146

Table 2
The values of K_j^*'s in Model B

j	K_j^*		
	$\alpha_\delta = -\alpha$	$\alpha_\delta = 0$	$\alpha_\delta = \alpha$
1	0.488350	0.502480	0.515885
2	0.485891	0.500334	0.514024
3	0.485775	0.500486	0.514415
4	0.487146	0.502093	0.516230
5	0.489513	0.504673	0.518995
6	0.492572	0.507926	0.522415
7	0.496126	0.511658	0.526297
8	0.500037	0.515733	0.530509
9	0.504209	0.520056	0.534956
10	0.508571	0.524556	0.539570
11	0.513068	0.529181	0.544296
12	0.517658	0.533888	0.549095

Table 3
A comparison of the results of the proposed models

α_δ	(n^*, TC^*)		
	Model A	Model A1	Model B
$-\alpha$	(13, 17177.92)	(13, 17177.20)	(12, 17103.30)
0	(13, 17198.94)	(13, 17198.20)	(12, 17120.28)
$+\alpha$	(13, 17219.14)	(13, 17218.30)	(12, 17136.28)

Table 4
Effects of amelioration on the total inventory cost

α	Total cost (TC^*)		
	Model A	Model A1	Model B
0.0	17198.94	17198.20	17120.28
0.01	17177.92	17177.20	17103.30
	17178.02#		17103.83#
0.03	17132.55	17131.94	17066.02
	17136.37#		17071.06#
0.05	17078.45	17077.98	17020.16
	17095.00#		17038.52#

Example 2. We take $r = 0.1$, $i_1 = 0.06$, $i_2 = 0.08$ and keep the other parameter values same as in Example 1. In this example, we have for ameliorating items, $R_1 - \alpha > 0$, $R_2 - \alpha < 0$ when $\alpha = 0.03$ and $R_1 - \alpha < 0$, $R_2 - \alpha < 0$ when $\alpha = 0.05$. In both the situations, the optimality and uniqueness of the solution have been verified. For deteriorating items, the total costs and the number of replenishments are found in increasing trend with α whereas for ameliorating items those are in decreasing trend. The results are shown in Tables 5 and 6.

It is clear from the numerical study (Tables 4 and 6) that failure to include an ameliorating factor in inventory may result in significant additional cost.

Table 5
Impacts of α on the optimal results for deteriorating inventory ($\alpha_\delta = \alpha$)

α	(n^*, TC^*)		
	Model A	Model A1	Model B
0.0	(12, 21597.98)	(12, 21597.94)	(12, 21539.80)
0.01	(13, 21636.96)	(13, 21636.91)	(12, 21573.32)
0.03	(13, 21706.24)	(13, 21706.17)	(12, 21635.26)
0.05	(14, 21768.43)	(14, 21768.36)	(13, 21688.98)

Table 6
Impacts of α on the optimal results for ameliorating inventory ($\alpha_\delta = -\alpha$)

α	(n^*, TC^*)		
	Model A	Model A1	Model B
0.0	(12, 21597.98)	(12, 21597.94)	(12, 21539.80)
0.01	(12, 21556.17) 21556.94#	(12, 21556.13)	(11, 21501.93) 21503.03#
0.03	(11, 21463.49) 21475.36#	(11, 21463.47)	(11, 21418.05) 21428.43#
0.05	(10, 21353.81) 21394.43#	(10, 21353.80)	(10, 21315.88) 21354.40#

5. Conclusion

In reality the value or utility of goods, while in stock, may decrease (in case of deteriorating items) or increase (in case of ameliorating items) over time. In this paper, inventory models have been developed considering both the opposite physical characteristics (amelioration and deterioration) of stored items. The study has been conducted under the Discounted Cash Flow (DCF) approach as it permits a proper recognition of the financial implication of the opportunity cost in inventory analysis. As the goal of the paper is to incorporate the amelioration and deterioration phenomenon together into an inventory model over a finite planning horizon, we have restricted our study in the simplest inventory ordering policy viz. the equal order-interval policy. Of course, better results could be obtained upon relaxation of this restriction.

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